

# Decoherence in adiabatic quantum computation

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We have studied the decoherence properties of adiabatic quantum computation in the presence of in general non-Markovian (e.g., low-frequency) noise. We show that the global scheme of adiabatic quantum computation maintains its performance even for strong decoherence. The more efficient local adiabatic computation, however, does not improve scaling of the computation time with the number of qubits  $n$  as in the decoherence-free case, although it does provide some “prefactor” improvement. The scaling improvement requires phase coherence throughout the computation, limiting the computation time and the problem size  $n$ .

The possibility of using quantum mechanics for computation has recently gained considerable interest [1, 2]. Although the gate model of quantum computation (QC) is theoretically well developed, the necessity of preserving quantum coherence and complex patterns of entanglement for a long period of time, as well as the extreme precision requirement for gate operations makes its implementation difficult. Quantum error correcting codes [3–6] and fault-tolerant implementations [7, 8] enable QC with finite precision and in the presence of external noise, although with significant overhead.

Recently, other computation protocols have been suggested which may require less quantum resources. One of these is adiabatic quantum computation (AQC) [9], which is known to be computationally equivalent to the gate model of QC [10]. In AQC, a simple initial Hamiltonian  $H_i$  is slowly deformed into a final Hamiltonian  $H_f$ :  $H_S = [1 - s(t)]H_i + s(t)H_f$ , with  $s(t)$  changing from 0 to 1 between the initial ( $t_i=0$ ) and final ( $t_f$ ) times. The qubit register starts from the ground state of  $H_i$ , into which it is designed to be initialized with high fidelity. If the evolution is slow enough (adiabatic), the system ends up in the ground state of  $H_f$  which encodes the solution to a computationally interesting problem.

The performance of the algorithm is limited by the regions of  $s$  close to energy anticrossings, where the energy gap  $g(s)$  between the first two energy levels reaches the minimum  $g_m \equiv \min_s[g(s)]$ . Assuming that the minimum gap occurs at  $s=s_m$ , it is convenient to adopt a new coordinate,  $\epsilon = E(s - s_m)$ , where  $E \gg g_m$  is an energy scale characterizing the anticrossing. If  $g_m$  is much smaller than the separation of the two crossing levels from other energy levels, then the slow evolution of the system close to the anticrossing will be restricted only to those levels. In this region, the gap between the two lowest states is well approximated by  $g = \sqrt{\epsilon^2 + g_m^2}$ , and the system Hamiltonian in such a 2-state subspace is

$$H_S = -(\epsilon\sigma_z + g_m\sigma_x)/2, \quad (1)$$

where  $\sigma$ 's are the Pauli matrices. If the evolution satisfies the adiabatic condition ( $\hbar=k_B=1$ ):  $|\langle 1|dH/d\epsilon|0\rangle|\dot{\epsilon} \ll g^2(s)$ , then at  $t = t_f$ , the system will be in the ground

state of  $H_f$  with probability close to one, and the solution to the problem may then be read out.

The “global” adiabatic scheme is based on the linear interpolation,  $s = t/t_f$ , so that  $\dot{s}$  is a constant, and the adiabatic condition must be satisfied for the smallest gap. This yields  $\dot{\epsilon} = \alpha g_m^2$ , where  $\alpha$  is a small dimensionless number. The computation time then is

$$t_f = E/\alpha g_m^2 \equiv t_{\text{global}}, \quad (2)$$

while the time the system spends near the anticrossing,  $\sim (g_m/E)t_f$ , is only a small fraction of the total time (2).

In the “local” scheme [11], the time variation of  $s$  is chosen so that the system spends the main part of the adiabatic evolution in the vicinity of the anticrossing. This is achieved by satisfying the adiabatic condition uniformly, i.e., taking  $\dot{\epsilon}(t) = \alpha g(t)^2$ . Integration of this condition near the anticrossing gives  $\epsilon(t) = g_m \tan(\alpha g_m t - \pi/2)$ , and the computation time is

$$t_f = \pi/\alpha g_m \equiv t_{\text{local}}. \quad (3)$$

Such a scheme is optimal [11] meaning that no other  $s(t)$  can provide shorter computation time. In general, finding  $g(s)$  is as hard as solving the original problem, and therefore the assumption that it is known *a priori* is not always reasonable. In some special cases, however, such as the adiabatic Grover search [11],  $g(s)$  is independent of the final solution and can be calculated analytically. Hamiltonians with this property are usually projective and hard to implement due to  $n$ -local interactions.

The algorithm performance in both cases is determined by  $g_m$ . As the size of the problem and therefore the number of qubits  $n$  grows,  $g_m$  shrinks, requiring increasingly longer  $t_f$ . Clearly, the local adiabatic evolution shows a better scaling, with  $t_{\text{local}} \propto g_m^{-1}$  as opposed to  $t_{\text{global}} \propto g_m^{-2}$  in the global case.

The local adiabatic evolution plays a crucial role in the scaling analysis of the AQC [11–13], even when the assumption of local scheme is made only implicitly. E.g., suppose a lower bound  $O[f(n)]$  is obtained for the evolution time of  $n$  qubits in the case of general  $s(t)$ . Such a bound applies only to the *optimal*, i.e. local, scheme. For a more practical global evolution, the real lower bound

is then  $O[f(n)^2]$ , which in some cases coincides with the classical performance.

The enhanced performance of the local scheme comes at a price of its stronger sensitivity to decoherence. As demonstrated in this work, the coupling of the qubit register to an external dissipative environment has different effects on both schemes. While the global computation time (2) is essentially unaffected even by the strong decoherence which broadens the anticrossing region to a width  $W \gg g_m$ , the time (3) is increased significantly in the regime of strong decoherence, essentially eliminating the advantage of the local scheme. A qualitative reason for this is that the low-frequency component of the environmental noise introduces uncertainty in the position of the anticrossing point, making it necessary to slow down the adiabatic evolution for a longer period of time. Below we study quantitatively the anticrossing broadening and the resulting computation times for different environments, focusing mostly on the regime of large temperature  $T$  and strong dissipation,  $g_m \ll T, W$ , which can also be viewed as the limit of the large-scale “difficult” problems. AQC can still function at such large temperatures  $T$  in the “probabilistic” mode, since the probability to stay in the ground state reduces only to 1/2 if  $T$  is still smaller than the larger gaps of the algorithm Hamiltonian [17].

Decoherence is introduced as usual by adding the bath  $H_B$  and the interaction Hamiltonian  $H_{\text{int}}$  to the Hamiltonian  $H_S$  of the qubit register:  $H_{\text{total}} = H_S + H_B + H_{\text{int}}$ . In general, any single-qubit error excites the system out of the subspace (1) of the two lowest levels. Near the anticrossing, the gaps to higher levels should be much larger than  $g_m$ , and these processes are suppressed, so that both  $H_S$  and  $H_{\text{int}}$  are reduced to the 2-state form, and given, respectively, by (1) and  $H_{\text{int}} = -Q\sigma_z$ , where  $Q$  is an operator of the environmental noise.

In the case of a *bosonic environment*, the bath is modeled as an ensemble of harmonic oscillators distributed over frequencies  $\omega$ ,  $H_B = \sum_{\omega} \omega a_{\omega}^{\dagger} a_{\omega}$ , and the environmental force  $Q$  can be taken in the form  $Q = \sum_{\omega} \lambda_{\omega} (a_{\omega}^{\dagger} + a_{\omega})$ . The coupling constants  $\lambda_{\omega}$  determine the strength of the decoherence through the noise correlator  $S(\omega) = J(\omega)/(1 - e^{-\omega/T})$ , where  $J(\omega) = 2\pi \sum_{\omega', \pm} \lambda_{\omega'}^2 \delta(\omega \pm \omega')$ .

Decoherence can be viewed as a result of the bath-qubit entanglement produced by  $H_{\text{int}}$ : The shift of the bath oscillators from their positions depends on the state of the qubit register in the  $\sigma_z$  basis. In the case of strong decoherence considered here, such shifts localize the  $\sigma_z$  states, and the tunnel coupling represented by  $g_m$  in (1) can be treated as a perturbation – see, e.g., [14]. In the lowest order in  $g_m$ , the rate of tunneling between the two  $\sigma_z$  basis states is:

$$\Gamma(\epsilon) = \frac{g_m^2}{4} \int dt e^{i\epsilon t} \exp \left\{ \frac{2}{\pi} \int \frac{d\omega}{\omega^2} S(\omega) (e^{-i\omega t} - 1) \right\}. \quad (4)$$

Since the characteristic time  $t_f$  of the adiabatic evolution is much larger than the time scale  $1/\epsilon$  in (4), the

overall change of the  $z$ -component of the density matrix  $\rho$  in the  $\sigma_z$  basis can be obtained by integrating the rate equation for the balance of transitions (4) between the  $\sigma_z$  states over the evolution of  $\epsilon(t)$ :

$$\frac{\rho_z(t_f)}{\rho_z(0)} = e^{-\bar{\Gamma} t_f}, \quad \bar{\Gamma} \equiv \frac{2}{t_f} \int_{-\infty}^{\infty} \Gamma(\epsilon) \frac{d\epsilon}{\dot{\epsilon}}. \quad (5)$$

The condition of proper functioning of the adiabatic algorithm is that the evolution time  $t_f$  is sufficiently long on the scale of the average transition rate  $\bar{\Gamma}$ , so that the system which starts in the ground state,  $\rho_z(0) = 1$ , after the anticrossing is still found in the ground state with appreciable probability  $p = (1 - e^{-\bar{\Gamma} t_f})/2$ . One immediate consequence of Eqs. (4) and (5) is that for the global adiabatic evolution,  $\dot{\epsilon} = \text{const} = E/t_f$ , we have  $\bar{\Gamma} = \pi g_m^2/E$ , independently of  $S(\omega)$ , and the required computational time  $t_f \simeq \bar{\Gamma}^{-1}$  coincides with the decoherence-free case (2). [If one allows the same probabilistic operation of the algorithm as in the strong-decoherence regime, then  $\alpha \simeq 1$  in Eq. (2).] This result agrees with conclusions of [15–17] for Landau-Zener transitions with an ohmic bath.

In contrast to the global evolution, characteristics of the local adiabatic scheme are sensitive to the energy dependence of the tunneling rate (4) and therefore to the environment spectrum  $S(\omega)$ . We assume first the model  $S(\omega) = \frac{\gamma}{2} [1 + (\omega/\omega_c)^2]^{-1}$ , that depending on the value of the cut-off energy  $\omega_c$ , allows for uniform description of the high- $T$  ohmic dissipation (white noise) and the low-frequency noise [18] typical for most solid-state qubits – see, e.g., [19]. Substitution into (4) gives:

$$\begin{aligned} \Gamma(\epsilon) &= \frac{g_m^2}{4} \int dt e^{i\epsilon t} \exp \{ -\gamma (|t| + (e^{-\omega_c |t|} - 1)/\omega_c) \} \\ &= \frac{g_m^2}{2} \cdot \begin{cases} \gamma / (\epsilon^2 + \gamma^2), & \omega_c \gg \gamma, \\ (\pi/2\gamma\omega_c)^{1/2} e^{-\epsilon^2/2\gamma\omega_c}, & \omega_c \ll \gamma, \end{cases} \quad (6) \end{aligned}$$

The width of the “resonance” region, where  $\Gamma$  is nonvanishing, is

$$W = \begin{cases} \gamma, & \omega_c \gg \gamma, \\ (\gamma\omega_c)^{1/2}, & \omega_c \ll \gamma, \end{cases} \quad (7)$$

which under our assumptions is much larger than  $g_m$ .

In the case of local evolution,  $\dot{\epsilon} = \alpha g^2$ , the average tunneling rate in (5) is dominated by the vicinity of the point  $\epsilon = 0$ . Quantitatively, (3) means that  $(t_f \dot{\epsilon})^{-1} \approx \delta(\epsilon)$ , and (5) and (6) give:  $\bar{\Gamma} = 2\Gamma(0) \sim g_m^2/W$ . Thus, for the computation time  $t_f \simeq \bar{\Gamma}^{-1}$  we have:

$$t_f \sim (W/E) \cdot t_{\text{global}} \propto g_m^{-2}, \quad (8)$$

and besides the prefactor enhancement of  $W/E$  over the global scheme, no scaling improvements achieved by using the local adiabatic evolution.

It is instructive to note that the large- $\omega_c$  limit of (6) coincides with the tunneling rate that can be obtained

directly from the standard Bloch-Redfield formalism as applied to strong  $\delta$ -correlated noise (see, e.g., [17, 20]). In contrast to this, the small- $\omega_c$  case in (6) can be understood as an effect of the quasi-static thermal fluctuations creating an uncertainty in the sweep parameter  $\epsilon$ . This result can be extended straightforwardly to an arbitrary low-frequency *classical noise*  $\delta\epsilon$  that is generated by some (not necessarily bosonic) environment. In the case of the global adiabatic evolution, such a noise just shifts the time axis without affecting the transition probabilities [24], so that the same time scale (2) is obtained for the algorithm with or without noise. By contrast, the local adiabatic protocol is strongly affected by the noise with large amplitude,  $W \equiv (\overline{\delta\epsilon^2})^{1/2} \gg g_m$ . Indeed, if one takes  $\dot{\epsilon} = \alpha g^2$  as without the noise, the slow-down of the evolution will most of the times miss the actual anticrossing, and the probability  $p$  of staying in the ground state will be suppressed as  $g_m^2/W^2$ . To increase this probability to  $p \simeq 1/2$  by reducing  $\alpha$ , one needs to make it so small that the computation time scales even worse than in the global case,  $t_f \simeq W^2/g_m^3$ . If an estimate for  $W$  is known, a better strategy is to slow down the evolution in a region of width  $W$  around  $\epsilon = 0$ , which yields (8), i.e., a prefactor enhancement.

This role of the low-frequency noise suggests that it might be useful to separate this noise from the relaxation transitions induced by the high-frequency parts of  $S(\omega)$ . To do this quantitatively we consider an environment with removed low-frequency part:  $S(\omega) = \frac{\gamma}{2} \{ [1 + (\omega/\omega_c)^2]^{-1} - [1 + (\omega/\omega_0)^2]^{-1} \}$  with  $\omega_0 \ll \omega_c$ . In the context of superconducting qubits, such a “frequency-dependent” relaxation is used experimentally [21, 22] for the same purpose of suppression of the low-frequency noise. The tunneling rate is obtained then as before:

$$\Gamma(\epsilon) = \frac{g_m^2}{4} \int dt e^{i\epsilon t} \exp\left\{\gamma \left[ \frac{1 - e^{-\omega_c|t|}}{\omega_c} - \frac{1 - e^{-\omega_0|t|}}{\omega_0} \right]\right\}.$$

At low energies  $\epsilon \ll \omega_c$ , the  $\omega_c$ -part can be neglected:

$$\Gamma(\epsilon) = \frac{g_m^2 e^{-\gamma/\omega_0}}{2} \left[ \pi \delta(\epsilon) + \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\gamma(\gamma/\omega_0)^n}{\epsilon^2 + [(n+1)\omega_0]^2} \right].$$

The first,  $\delta$ -function, term describes the “dissipation-free” transition rate but with renormalized tunnel coupling  $g_m \rightarrow g_m^* = g_m e^{-\gamma/2\omega_0}$ . In our approach, that is perturbative in  $g_m$ , this term has the  $\delta$ -functional dependence on energy. Although in the more accurate treatment this term should actually be broadened by the tunnel coupling  $g_m^*$  itself, this broadening does not change the integral transition probability. The sum over  $n$  represents the dissipative tunneling. Taken as a whole, this equation describes the crossover from the dissipation-free tunneling in the case  $\gamma \ll \omega_0$ , when the environmental noise is removed from all relevant frequencies  $\simeq \gamma$ , to the dissipation-dominated tunneling rate described by

the first part of Eq. (6) for  $\gamma \gg \omega_0$ , when the modification of environmental noise at frequencies  $\simeq \omega_0$  becomes irrelevant. We see that removal of the noise at frequencies up to the relaxation rate  $\gamma$  suppresses the environmental broadening of the Landau-Zener transition. Similarly to the case of superohmic environment [17], the “pure” relaxation remaining in this case at larger energies can improve the algorithm performance (compared with the one provided by the renormalized gap  $g_m^*$ ). However, since the relaxation rate (6) scales at large energies  $\epsilon > \omega_0$  as  $g_m^2 \gamma/\epsilon^2$ , this prefactor improvement does not change the scaling of the algorithm performance with the gap  $g_m$ .

So far we have discussed the large-temperature regime  $T \gg W, g_m$ , when the environmental noise is effectively classical. We now consider briefly the case of low temperatures  $T \ll g_m, W$  and show that, qualitatively, the quantum low-frequency noise has the same effect of broadening the Landau-Zener transition and thus changing the scaling of local adiabatic algorithms. Quantitative characteristics of the quantum noise are, however, quite different. The main difference is that besides the broadening of the transition resonance, the quantum noise leads also to the “polaron” shift of the resonance. While the classical broadening is symmetric around the point  $\epsilon = 0$ :  $\Gamma(-\epsilon) = \Gamma(\epsilon)$ , in the quantum case, the resonance is shifted to positive energies.

To show this explicitly, we assume the simplest case of the single-mode environment of frequency  $\Omega$ :  $S(\omega) = 2\pi\lambda^2\delta(\omega - \Omega)$  and negligible temperature  $T$ . In this case, the tunneling rate (4) is:

$$\Gamma(\epsilon) = \frac{\pi g_m^2}{2} e^{-(2\lambda/\Omega)^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2\lambda}{\Omega}\right)^{2n} \delta(\epsilon - n\Omega). \quad (9)$$

We see that the environment splits the level crossing at  $\epsilon = 0$  into a sequence of crossings at  $\epsilon = n\Omega$ , corresponding to  $n$  photon emission process, with intensity dependent on the coupling strength  $\lambda$ . Integrating Eq. (9) over the evolution of  $\epsilon(t)$  similarly to Eq. (5) we obtain the “Landau-Zener” probability  $q$  for the system to remain in the initial state of the adiabatic dynamics not making the transition to the ground state:

$$-\ln q = \int_{-\infty}^{\infty} \Gamma(\epsilon) \frac{d\epsilon}{\dot{\epsilon}} = \frac{\pi g_m^2 e^{-(2\lambda/\Omega)^2}}{2} \sum_{n=0}^{\infty} \frac{(2\lambda/\Omega)^{2n}}{\dot{\epsilon}_n n!}. \quad (10)$$

Here  $\dot{\epsilon}_n$  is the rate of evolution of  $\epsilon$  at the  $n$ th crossing. For uniform evolution  $\dot{\epsilon}_n = \text{const} \equiv v$ , Eq. (10) reduces to the regular Landau-Zener probability  $q = e^{-\pi g_m^2/2v}$ , which at  $T = 0$  is independent of the coupling to the environment [23]. In contrast to the total probability, the width of the transition region depends strongly on the coupling strength  $\lambda$ . The low-frequency environment,  $\Omega \ll \lambda$ , makes the coupling strong. In this case, the Poisson distribution of the effective tunnel amplitudes in Eq. (10) can be well approximated as a Gaussian,

with position  $\bar{\epsilon} = 4\lambda^2/\Omega$  and width  $W = 2\lambda$ . This means that in order to keep the excitation probability sufficiently small, the local adiabatic evolution should be slowed down around the new resonance energy  $\epsilon = \bar{\epsilon}$  in the energy interval of the width  $W$ , which coincides with the typical magnitude  $\langle \tilde{Q} \rangle$  of the quantum fluctuations of the environmental noise  $Q$ . Once again we obtain the prefactor enhancement (8).

Finally, we discuss a *spin environment* which can also be important for practical qubits (see, e.g., [25]) and is in general composed of 2-level systems that can be viewed as spins. Qualitative properties of such an environment are similar to the bosonic one discussed above. In particular, at  $T = 0$ , the Landau-Zener transition probability  $q$  is unaffected by the spin environment [26, 27]. Although at non-vanishing  $T$  ground state probability  $p$  is reduced by coupling to environment, there exists an upper bound for the excitation probability, which yields finite  $p$  even when the number  $m$  of spins is large [26]. As a result, the global AQC can be done with the same time scale as for the closed system. Spin environment broadens the energy levels by splitting each level into  $2^m$  levels, so that the anticrossing is also split into  $2^{2m}$  anticrossings at random positions. As before, since the total width of the broadened region  $W$  is independent of  $g_m$ , only a prefactor enhancement (8) is possible.

To summarize, we have studied the decoherence effects due to different types of (non-Markovian) classical and quantum environments, including bosonic and spin baths, on AQC. In all cases, the environment broadens the energy levels and the transition region to a width  $W$  independent of the minimum gap  $g_m$ . The global AQC remains unaffected by such a broadening, but the local one, in the small-gap/strong-decoherence regime  $W > g_m$ , only provides a prefactor improvement of the algorithm running time (8) in comparison to the global scheme, and therefore does not change the scaling of this time with  $g_m$  as it does in the decoherence-free case. Thus, a local AQC can only maintain its properties if  $W < g_m$ . Since the width of the level broadening is related to the decoherence rate,  $W \sim 1/\tau_{\text{decoh}}$ , and  $t_f \sim 1/g_m$  for the local scheme in the weak-decoherence regime, the computation time is limited by the decoherence  $t_f < \tau_{\text{decoh}}$  in the same way as in gate model QC. Therefore, in order to keep the advantageous scaling of local AQC, phase coherence should be preserved throughout the evolution as in the gate model. The expectation that AQC is insensitive to phase coherence only holds for the global scheme and does not apply to local AQC.

It should be emphasized that for problems that are solvable on a gate model QC in polynomial time, a universal AQC [10, 28] with global scheme, can also provide solution in polynomial time, but with no need for maintaining phase coherence. Adiabatic Grover search [11], on the other hand, requires local scheme and therefore phase coherence in order to provide any advantage over

classical computation.

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